Adaptive Sampling for Best Policy Identification in MDPs

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RL Theory seminars
May 11th, 2021

Outline

1. Introduction
2. Lower Bound
3. Upper bound of the characteristic time
4. Algorithm
5. Experiments
6. Conclusion

How many samples does it take to learn an optimal policy in RL ?

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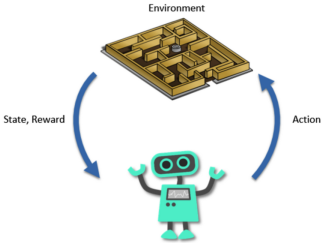

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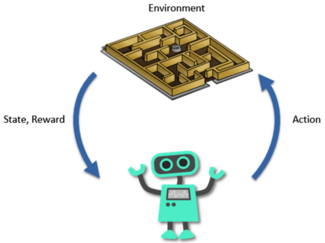
   - For simplicity, we assume q with support in $[0, 1]$.



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- **Assumption 1:** $\pi^\star \triangleq \pi^\star_\phi$ is unique.

Sampling schemes

- **Forward model:** The agent can only follow trajectories: $(s_0, a_0, R_0, s_1, a_1 \ldots,)$ where $s_{t+1} \sim p_\phi(.|s_t, a_t)$.

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In this talk, we focus on the Generative model.

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$\implies$ Algorithm with minimal *sample complexity* τ

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- Algorithms that sample state-actions uniformly at random are sufficient to be minimax optimal !

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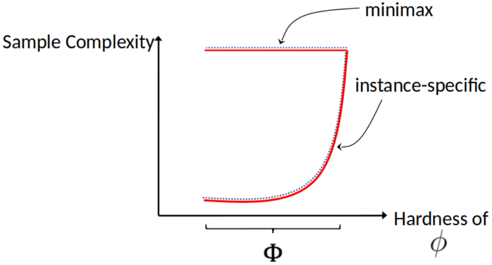

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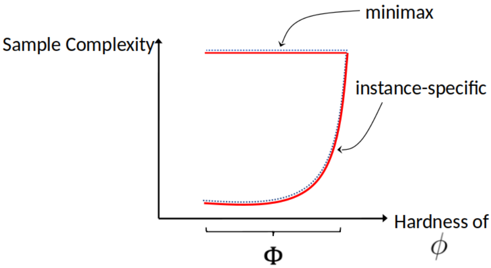

- We seek algorithms that can adapt to the hardness of the instance.

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     - [Garivier and Kaufmann, 2016] complete characterization for exponential family.
   - MDPs: [Zanette et al., 2019] proposed BESPOKE, first algorithm with problem-specific guarantees.

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Proposition 1

The sample complexity of any δ-PC algorithm satisfies: for any ϕ with a unique optimal policy,

$$\mathbb{E}_\phi[\tau] \geq T^\star(\phi) \log(1/2.4\delta),$$

$$\text{where } T^\star(\phi)^{-1} = \sup_{\omega \in \Sigma} \inf_{\psi \in \mathsf{Alt}(\phi)} \sum_{s,a} \omega_{sa} \mathrm{KL}_{\phi|\psi}(s,a). \tag{1}$$

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The set of alternative MDPs can be decomposed as follows:

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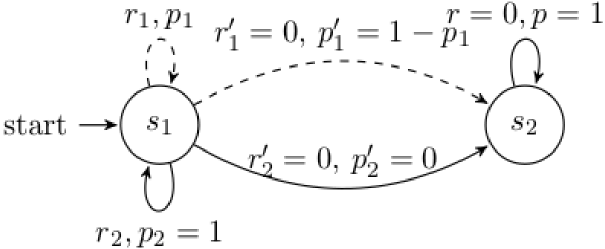
- Only involves $(r(s,a), p(s,a))$ and $\big(r(x,\pi^*(x)), p(x,\pi^*(x))\big)_{x\in\mathcal{S}}$ in ψ.

IT Lower bound: Intractable!



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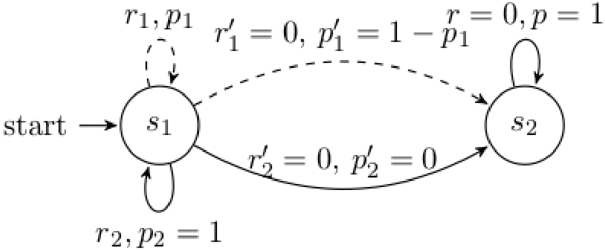

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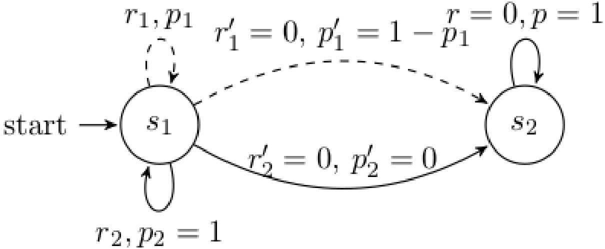

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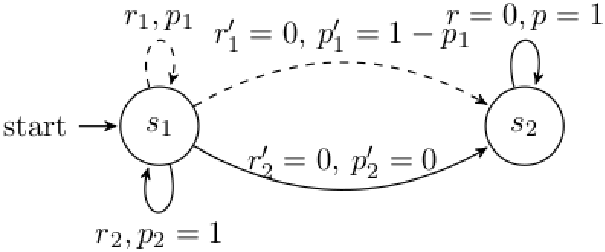

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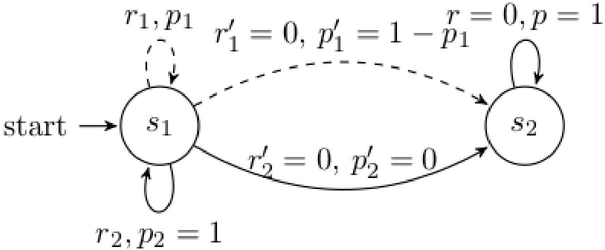

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  - $\phi = \frac{\psi+\overline{\psi}}{2}$ satisfies $\frac{r_1}{1-\gamma p_1} < \frac{r_2}{1-\gamma p_2}$.

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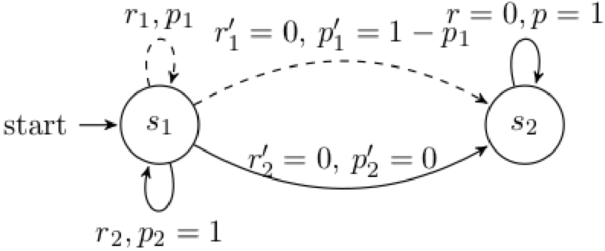

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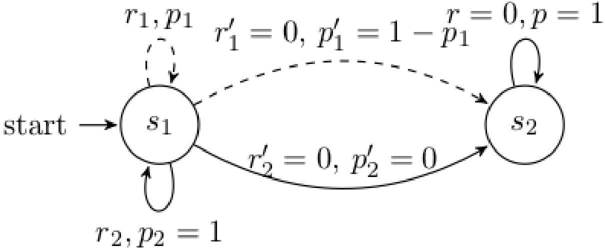

- Alt(ϕ) and $\mathrm{Alt}_{s_1 a_1}(\phi)$ are not convex.
- $\implies$ The sub-problem $\inf\limits_{\psi \in \mathrm{Alt}(\phi)} \sum\limits_{s,a} \omega_{sa} \mathrm{KL}_{\phi|\psi}(s,a)$ is non-convex.

IT Lower bound: MDP vs MAB

	MAB	**MDP**
Parameters	$\mu_1 > \ldots \geq \mu_K$	$\big(r(s,a), p(s,a)\big)_{s,a}$
Objective	Identify $a^\star = \arg\max_{a \in [K]} \ \mu_a$	Identify $\pi^\star = \arg\max_{\pi} \ (I - \gamma P_\pi)^{-1} r_\pi$
Alternative instances	$\bigcup_{a \neq 1} \{\lambda : \ \lambda_a > \lambda_1\}$ union of convex sets	$\bigcup_{(s,a \neq \pi^\star(s))} \{\psi : \ Q_\psi^{\pi^\star}(s,a) > V_\psi^{\pi^\star}(s)\}$ Not union of convex
IT lower bound	Tractable	Not Tractable

Upper bound: Idea

Define $T(\phi, \omega)^{-1} \triangleq \inf_{\psi \in \mathrm{Alt}(\phi)} \sum_{s,a} \omega_{sa} \mathrm{KL}_{\phi|\psi}(s, a)$.

Upper bound: Idea

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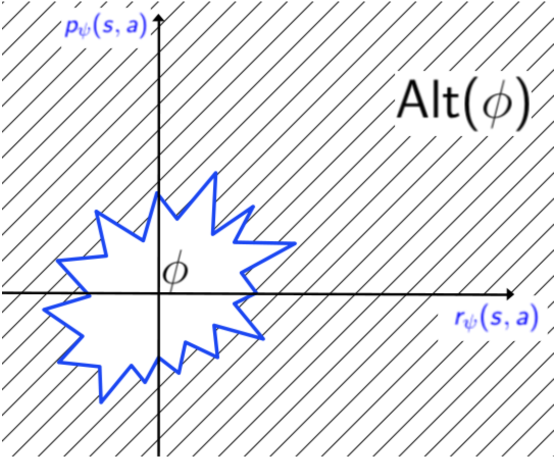

Figure: Alt(ϕ): Non-convex boundary

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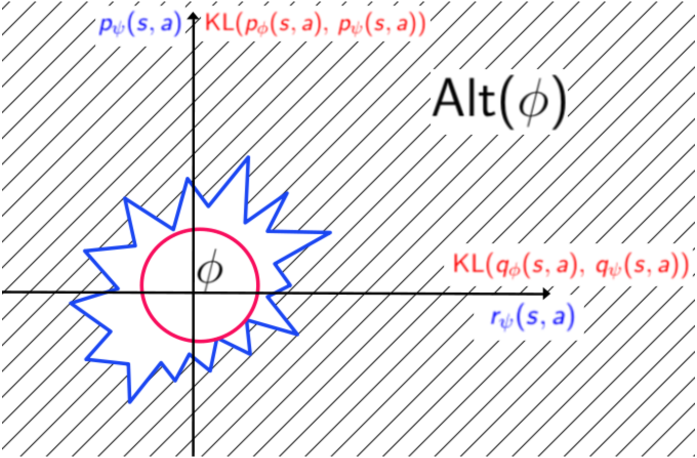

Figure: KL Ball

Upper bound of the characteristic time

Theorem 1 (Upper bound of minimal sample complexity)

For all vectors ω in the simplex:

$$T(\phi,\omega) \le U(\phi,\omega) \triangleq \max_{s,a\neq\pi^\star(s)} \frac{T_1(s,a;\phi)+T_2(s,a;\phi)}{\omega_{sa}} + \frac{T_3(\phi)+T_4(\phi)}{\min_s \omega_{s,\pi^\star(s)}},$$

$$\text{where} \begin{cases} T_1(s,a;\phi) = \dfrac{2}{\Delta_{sa}^2}, \\ T_2(s,a;\phi) = \max\left(\dfrac{16\mathrm{Var}_{p_\phi(s,a)}[V_\phi^\star]}{\Delta_{sa}^2}, \dfrac{6\mathrm{osc}[V_\phi^\star]^{4/3}}{\Delta_{sa}^{4/3}}\right), \\ T_3(\phi) = \dfrac{2}{[\Delta_{\min}(\phi)(1-\gamma)]^2}, \\ T_4(\phi) \le \dfrac{27}{\Delta_{\min}(\phi)^2(1-\gamma)^3} = \mathcal{O}\left(\dfrac{\text{Minimax lower bound}}{SA}\right) \end{cases}$$

Upper bound of $T(\phi,\omega)$: sketch of the proof

- Using Lemma 2:

$$T(\phi,\omega)^{-1} = \min_{s,a\neq\pi^\star(s)} \inf_{\psi\in\mathrm{Alt}_{sa}(\phi)} \omega_{sa}\mathrm{KL}_{\phi|\psi}(s,a) + \sum_x \omega_{x,\pi^\star(x)}\mathrm{KL}_{\phi|\psi}(x,\pi^\star(x)).$$

where $\mathrm{Alt}_{sa}(\phi) \triangleq \left\{\psi : \; Q_{\psi}^{\pi^\star}(s,a) > V_{\psi}^{\pi^\star}(s)\right\}$.

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where $\mathrm{Alt}_{sa}(\phi) \triangleq \left\{\psi : \ Q_{\psi}^{\pi^\star}(s,a) > V_{\psi}^{\pi^\star}(s)\right\}.$

- Introduce the suboptimality gaps: $\Delta_{sa} \triangleq V_{\phi}^{\pi^\star}(s) - Q_{\phi}^{\pi^\star}(s,a)$:

$$(Q_{\psi}^{\pi^\star} - Q_{\phi}^{\pi^\star})(s,a) - (V_{\psi}^{\pi^\star} - V_{\phi}^{\pi^\star})(s) > \Delta_{sa}.$$

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- Rewrite the condition in terms of the differences in kernels:

$$dr(s,a) + \gamma dp(s,a)^\intercal V_{\phi}^{\pi^*} + \gamma[p_{\psi}(s,a) - \mathbb{1}(s)]dV^{\pi^*} > \Delta_{sa}$$

where
$dr(s,a) = r_{\psi}(s,a) - r_{\phi}(s,a), dV^{\pi^*}(s,a) = V_{\psi}^{\pi^*}(s,a) - V_{\phi}^{\pi^*}(s,a)$ etc

Upper bound of $T(\phi, \omega)$: sketch of the proof

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Upper bound of $T(\phi, \omega)$: sketch of the proof

- $dr(s,a) + \gamma dp(s,a)^\top V_\phi^{\pi^*} + \gamma[p_\psi(s,a) - \mathbb{1}(s)]dV^{\pi^*} > \Delta_{sa}$
- $dV^{\pi^*} = A + B$ where:

$$A \triangleq \left(I - \gamma P_\psi^{\pi^\star}\right)^{-1} \left[r_\psi^{\pi^\star} - r_\phi^{\pi^\star}\right].$$

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- $dr(s,a) = \alpha_1 \Delta_{sa},\ dp(s,a) = \alpha_2 \Delta_{sa},\ A = \alpha_3 \Delta_{sa},\ B = \alpha_4 \Delta_{sa}$, with $\sum_i \alpha_i > 1$

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- Use Pinsker inequality and transportation lemmas to relate dr, dp to $\mathrm{KL}(q_\phi,\ q_\psi), \mathrm{KL}(p_\phi,\ p_\psi)$:

$$\frac{1}{2}(\alpha_1 \Delta_{sa})^2 \leq KL(q_\phi(s,a), q_\psi(s,a)).$$

Upper bound: sketch of the proof

- Use IT inequalities to relate dr, dp to $\mathrm{KL}(q_\phi,\ q_\psi), \mathrm{KL}(p_\phi,\ p_\psi)$:

$$
\begin{aligned}
\frac{\alpha_1^2}{T_1} &\leq KL(q_\phi(.|s,a), q_\psi(s,a)). \\
\frac{\alpha_2^2}{T_2} &\leq KL(p_\phi(.|s,a), p_\psi(s,a)). \\
\frac{\alpha_3^2}{T_3} &\leq \max_s KL(q_\phi(.|s,\pi^*(s)), q_\psi(.|s,\pi^*(s))). \\
\frac{\alpha_4^2}{T_4} &\leq \max_s KL(p_\phi(s,\pi^*(s)), p_\psi(s,\pi^*(s))).
\end{aligned}
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- The optimal weights minimizing the upper-bound program:

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  - $(s_{t+1}, a_{t+1}) \in \underset{(s,a) \in \mathcal{S} \times \mathcal{A}}{\arg\max} \sum_{s=1}^{t} \tilde{\omega}_{sa}(\widehat{\phi}_s) - N_{sa}(t)$.

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- Ensures that $\mathbb{P}_\phi\left(\forall (s,a) \in \mathcal{S} \times \mathcal{A}, \quad \lim_{t\to\infty} \frac{N_{sa}(t)}{t} = \overline{\omega}_{s,a}(\phi)\right) = 1$.

KLB-TS: stopping rule

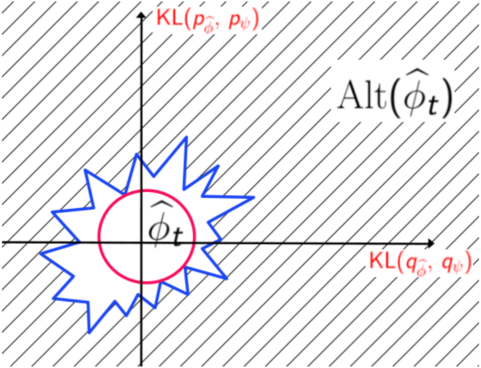

Figure: KL-Ball Stopping rule

KLB-TS: stopping rule

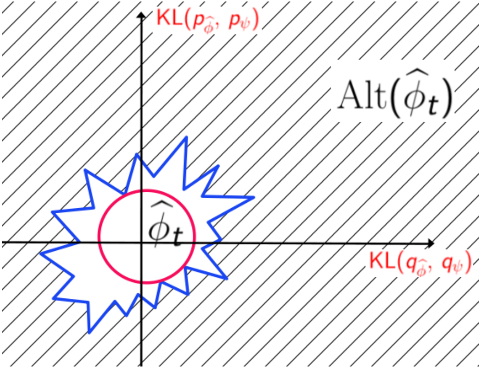

Figure: KL-Ball Stopping rule

- Need to ensure that ϕ falls within the KL-ball with probability $1 - \delta$.

Algorithm: Guarantees

Theorem 3

KLB-TS has a sample complexity τ_δ satisfying:
for all $\delta \in (0,1)$, $\mathbb{E}_\phi[\tau_\delta]$ is finite and $\limsup_{\delta \to 0} \frac{\mathbb{E}_\phi[\tau_\delta]}{\log(1/\delta)} \leq 4U(\phi)$, where:

$$
\begin{aligned}
U(\phi) &\triangleq \sup_{\omega} U(\phi, \omega) \\
&= \mathcal{O}\Bigg(S \max\left(\frac{\mathrm{Var}^\star_{\max}[V^\star_\phi]}{\Delta^2_{\min}(1-\gamma)^2}, \frac{1}{\Delta^2_{\min}(1-\gamma)^3} \right) \\
&\quad + \sum_{s,a \neq \pi^*(s)} \frac{1 + \mathrm{Var}_{p_\phi(s,a)}[V^\star_\phi]}{\Delta^2_{s,a}} \Bigg)
\end{aligned}
$$

Experiments

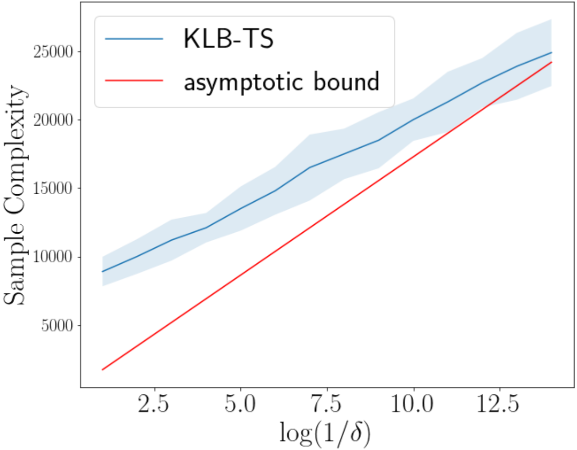

Figure: Asymptotic bound: S=A=2, $\gamma = 0.5$.

Experiments

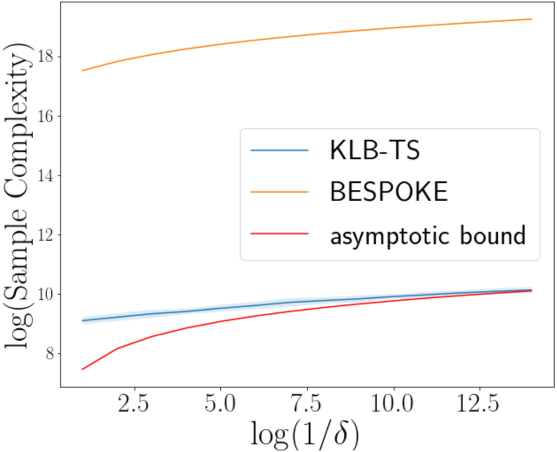

Figure: KLB-TS vs. BESPOKE. S=A=2, $\gamma = 0.5$.

Experiments

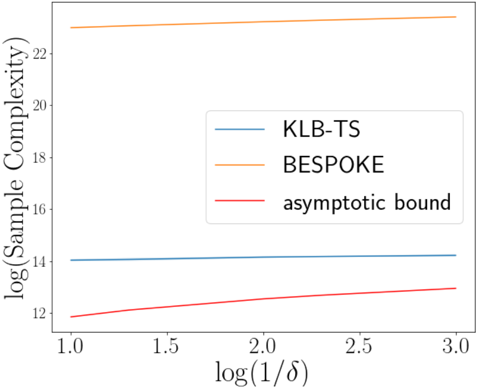

Figure: KLB-TS vs. BESPOKE. $S = 5, A = 10, \gamma = 0.7$.

- Most of BESPOKE's sample complexity comes from the burn-in phase $\Omega\left(\frac{S^2 A \log(1/\delta)}{(1-\gamma)^2}\right)$.

Conclusion

1. Algorithms designed using *problem-specific* bounds can achieve better sample complexity than minimax ones.
2. Contrary to MAB, IT lower bound is intractable for MDPs.
3. We can derive problem-specific surrogates which :
   - Are *explicit*, depending on functionals of the MDP.
   - Have a corresponding allocation that is easy to compute.
4. Can be used to devise (Asymptocically) Matching algorithm.
5. First step towards understanding problem-specific ε-optimal policy identification.

Thanks !

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